Particular Solutions

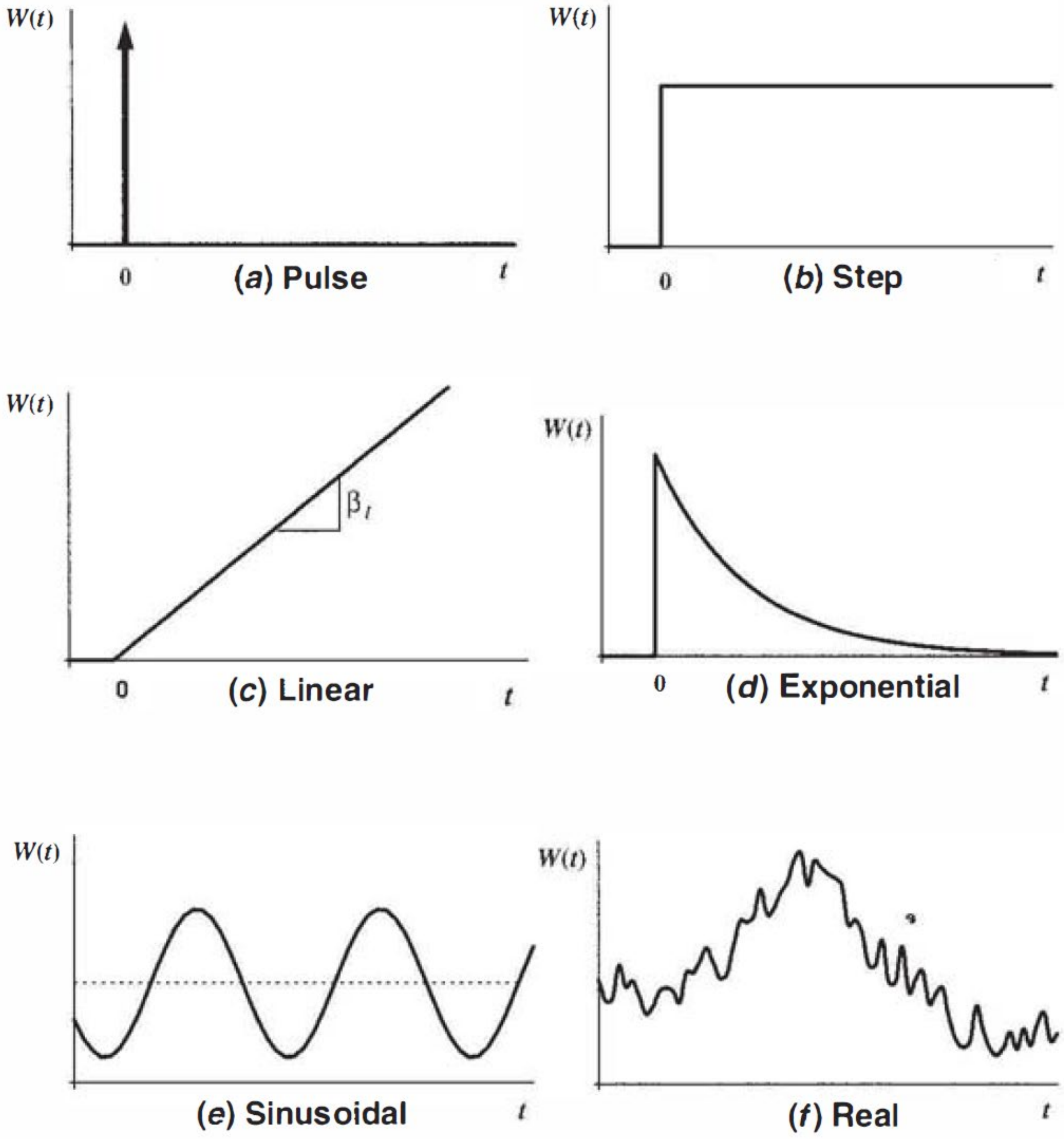


FIGURE 4.1
Loading functions $W(t)$ versus time t.

Impulse Loading (Spill)

Step Loading (New Cont. Source)

Linear ("Ramp") Loading

Exponential Loading

Sinusoidal Loading

The Total Solution: Linearity & Time Shifts

Fourier Series (Advanced Topics)

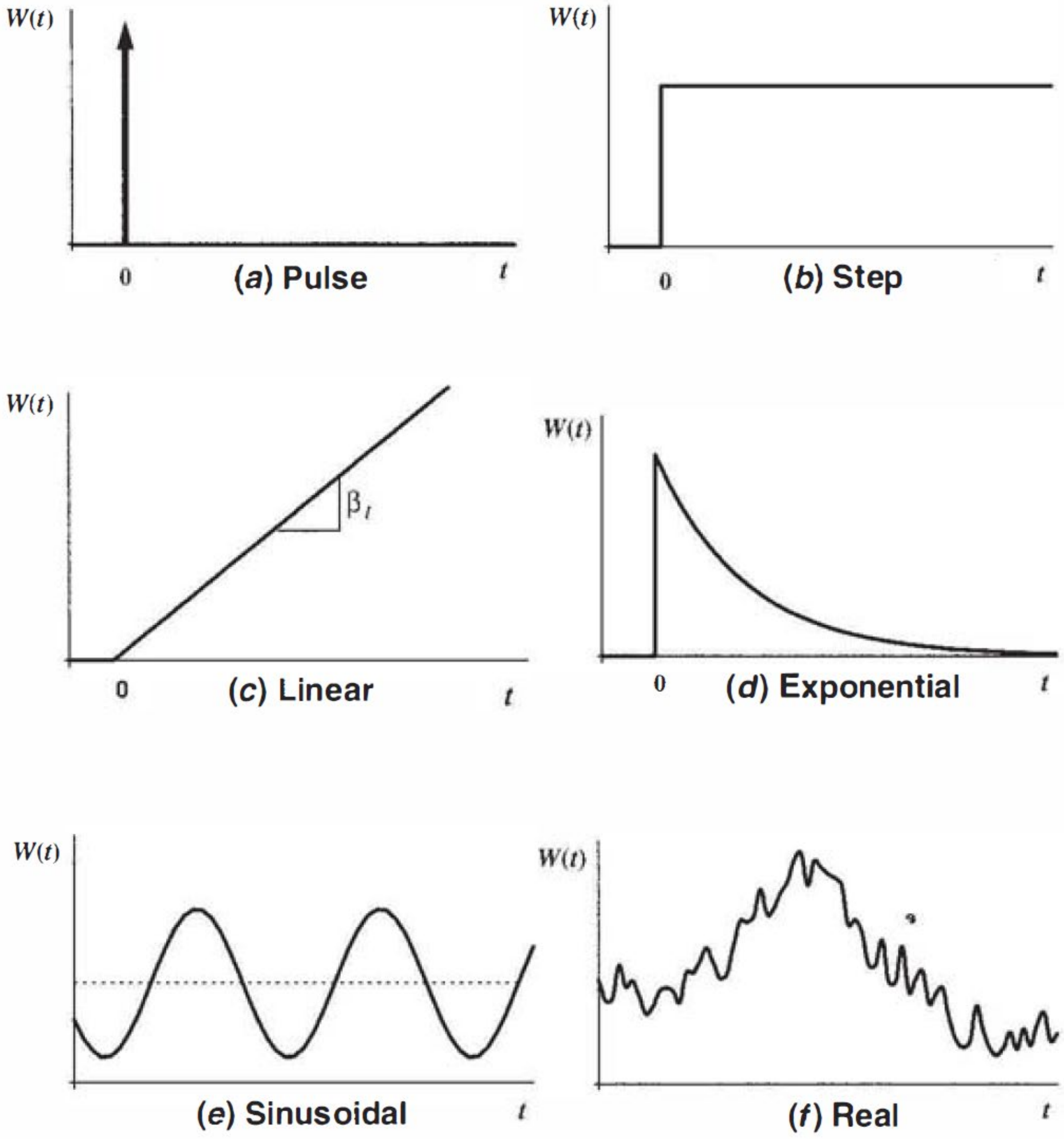


FIGURE 4.1
Loading functions $W(t)$ versus time t.

Particular Solutions

Up to now we derived the steady-state and general solutions for mass balance, CSTR, with first-order kinetics

$$\frac{dc}{dt} + \lambda c = \frac{W(t)}{V}$$

with λ being the eigenvalue which is:

$$\lambda = \frac{Q}{V} + k + \frac{v}{H}$$

Now we'll incorporate particular solutions of W(t) as depicted in idealized forms

←

Particular Solutions

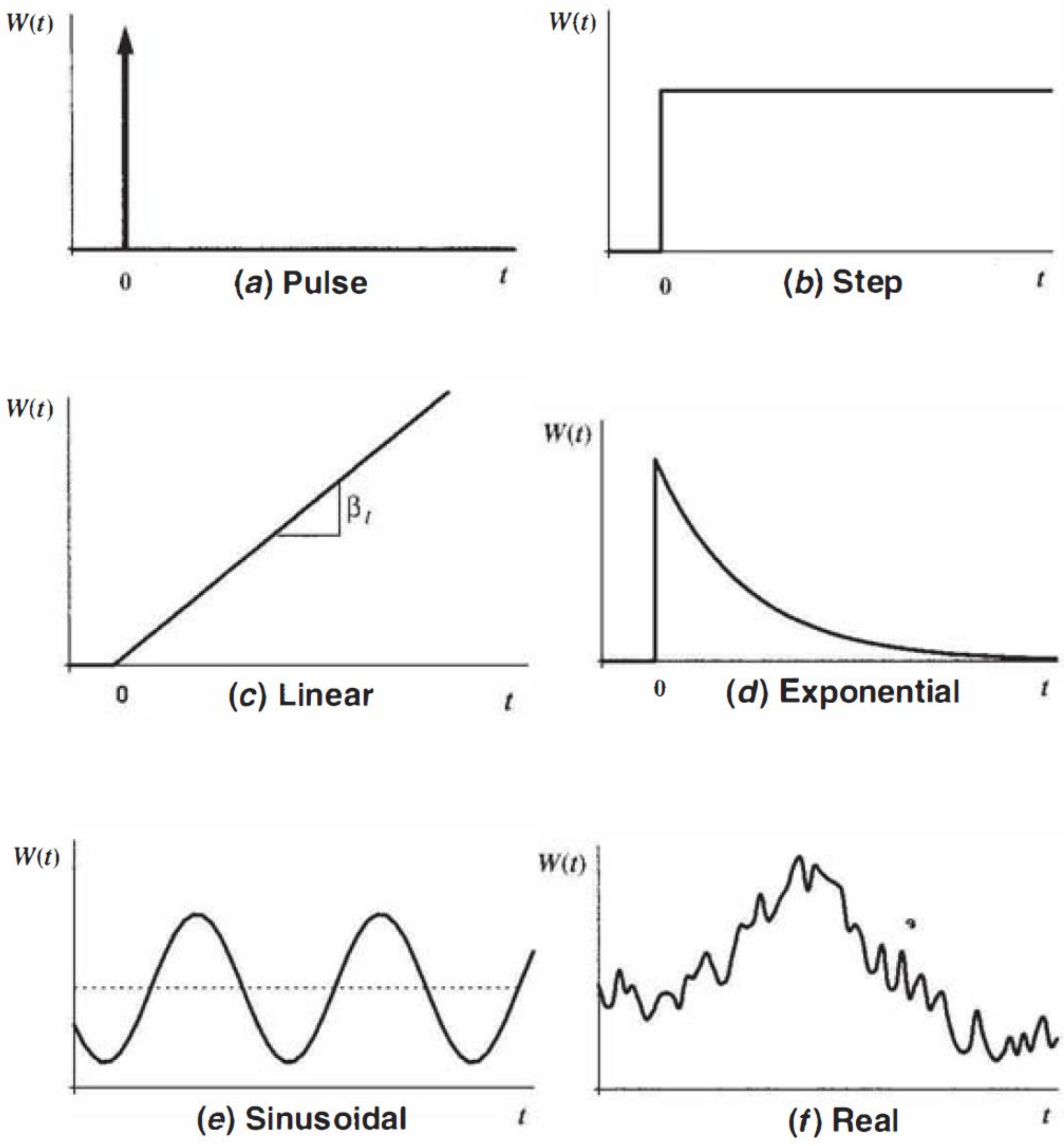


FIGURE 4.1
Loading functions $W(t)$ versus time t.

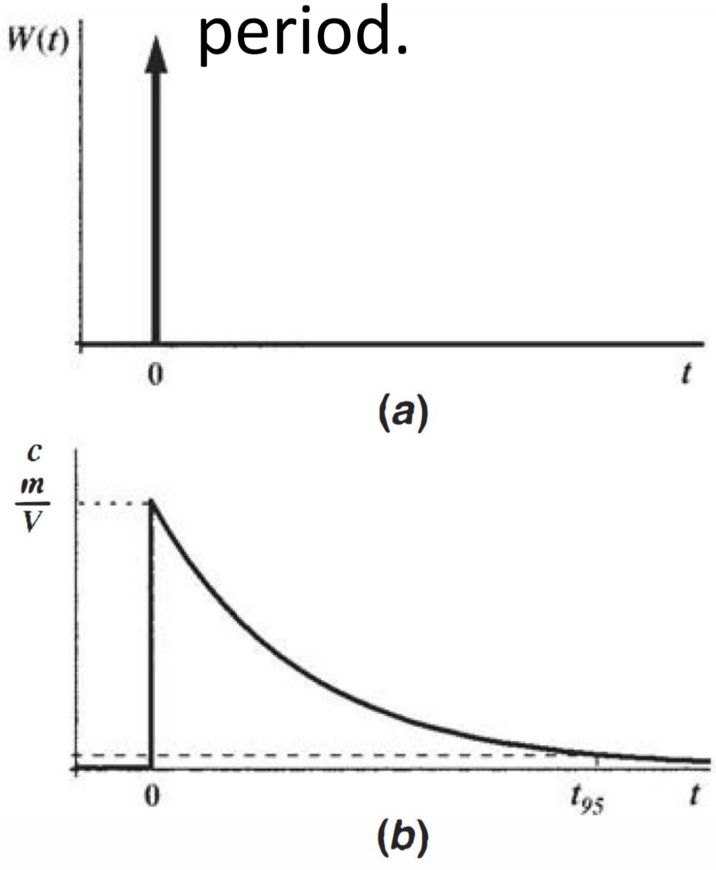
We'll show some **analytical solutions** for each of these loading functions.

Aside from mathematical representations we'll illustrate that each solution has a characteristic **shape parameters** that summarize behavior.

This focus is intended to build up mathematical insight regarding how natural waters respond to loadings.

Impulse Loading (Spill)

The most fundamental form of W(t) is the impulse loading representation of the discharge of waste over a relatively short time period.



The accidental spill of a contaminant to a water body demonstrates this well. Mathematically, the **Dirac delta function (δ(t))** represents such phenomena.

The delta function can be visualized as an infinitely thin spike at t=0.

FIGURE 4.2
Plot of (*a*) loading and (*b*) response for impulse loading.

Impulse Loading (Spill)

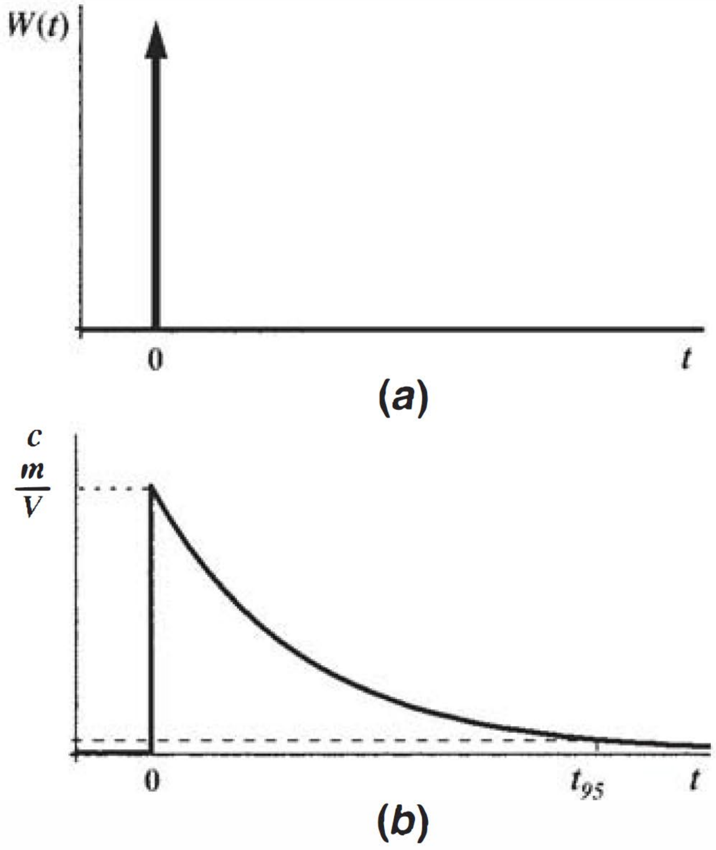


FIGURE 4.2
Plot of (*a*) loading and (*b*) response for impulse loading.

$$\delta(t) = 0 \qquad t \neq 0$$

and $$\int_{-\infty}^{\infty} \delta(t)dt = 1$$

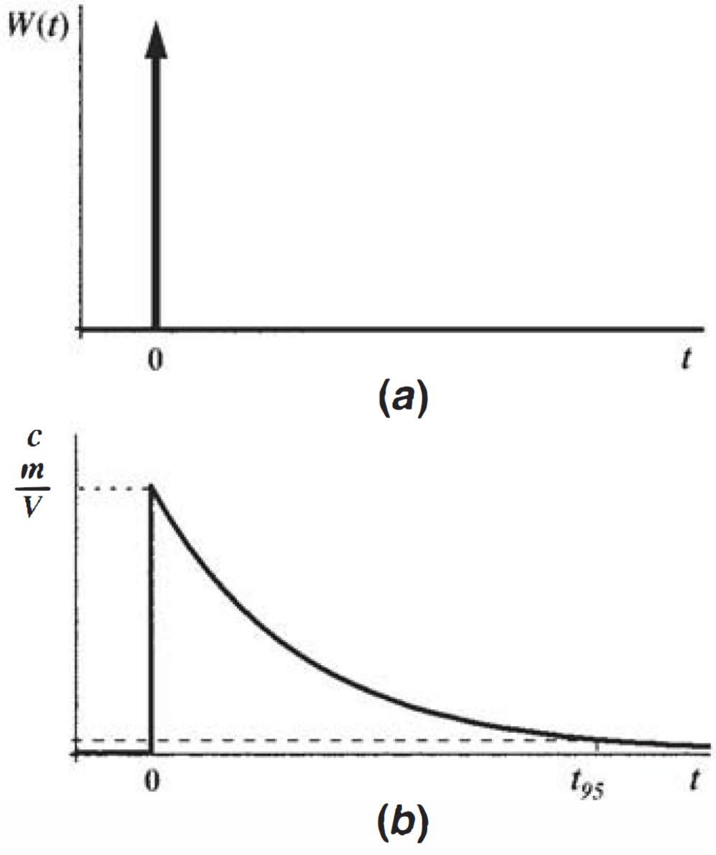
Impulse load of mass in terms of the delta function:

$$W(t) = m\delta(t)$$

$$\frac{dc}{dt} + \lambda c = \frac{m\delta(t)}{V}$$

Impulse Loading (Spill)

$$\frac{dc}{dt} + \lambda c = \frac{m\delta(t)}{V}$$



The particular solution for this case is:

$$c = \frac{m}{V} e^{-\lambda t}$$

This soln indicates instantaneous distribution and initial concentration of m/V, then decreases exp…

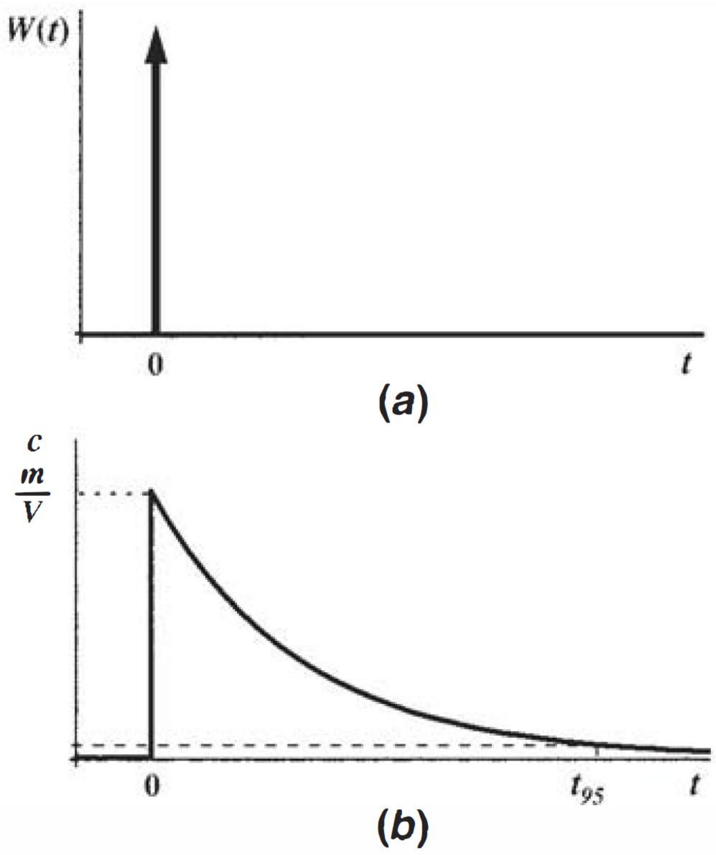
$$c_0 = \frac{m}{V}$$

FIGURE 4.2
Plot of (*a*) loading and (*b*) response for impulse loading.

Impulse Loading (Spill)

$$c_0 = \frac{m}{V}$$

Defines the first shape parameter (the height of the response) , and the response time (t_{φ}) defines its temporal extent. Given 95% response…:



$$t_{\varphi} = \frac{1}{\lambda} ln \frac{100}{100-\varphi}$$

$$t_{95} = \frac{1}{\lambda} ln \frac{100}{100-95} = \frac{3}{\lambda}$$

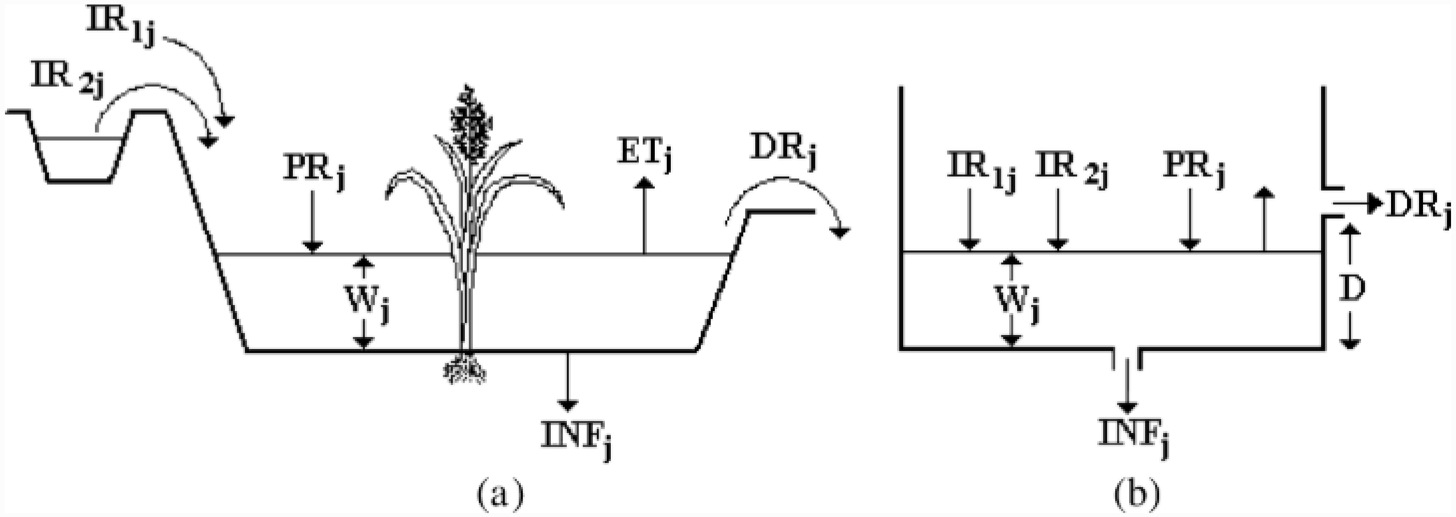
FIGURE 4.2
Plot of (*a*) loading and (*b*) response for impulse loading.

Model development for surface drainage loading estimates from paddy rice fields

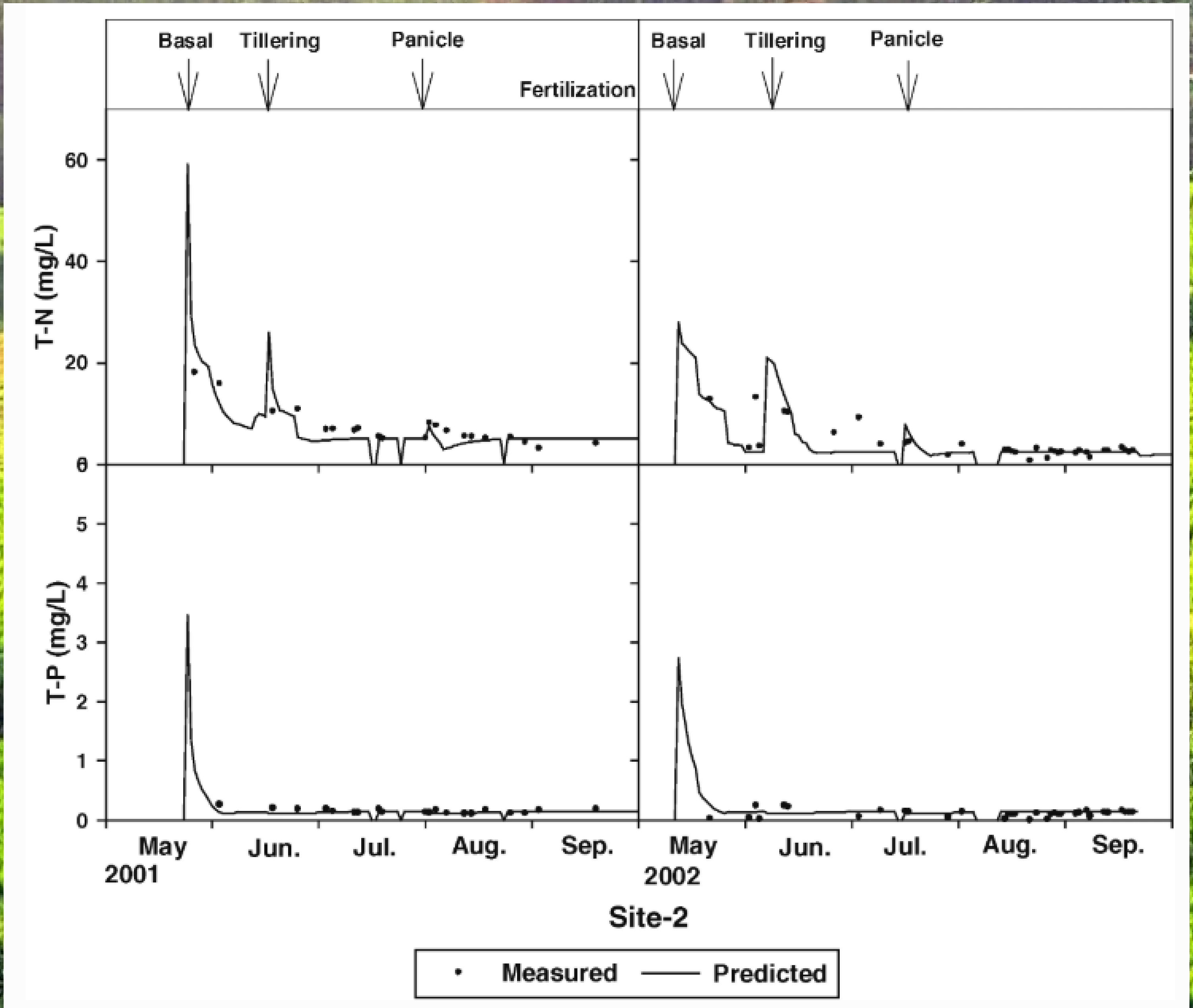
Ji-Hong Jeon, Chun G. Yoon, Jong-Hwa Ham & Kwang-Wook Jung

Paddy and Water Environment **3**, 93–101(2005) | Cite this article

$$W_j = W_{j-1} + \mathrm{IR}_{1j} + \mathrm{IR}_{2j} + \mathrm{PR}_j - (\mathrm{DR}_j + \mathrm{ET}_j + \mathrm{INF}_j)$$



Schematics of water balance components in paddy rice field (**a**) and model (**b**)



Basal
Tillering
Panicle
Fertilization
Basal
Tillering
Panicle
T-N (mg/L)
60
40
20
0
T-P (mg/L)
5
4
3
2
1
0
May
2001
Jun.
Jul.
Aug.
Sep.
May
2002
Jun.
Jul.
Aug.
Sep.
Site-2
Measured
Predicted

Step Loading (New Continuous Source)

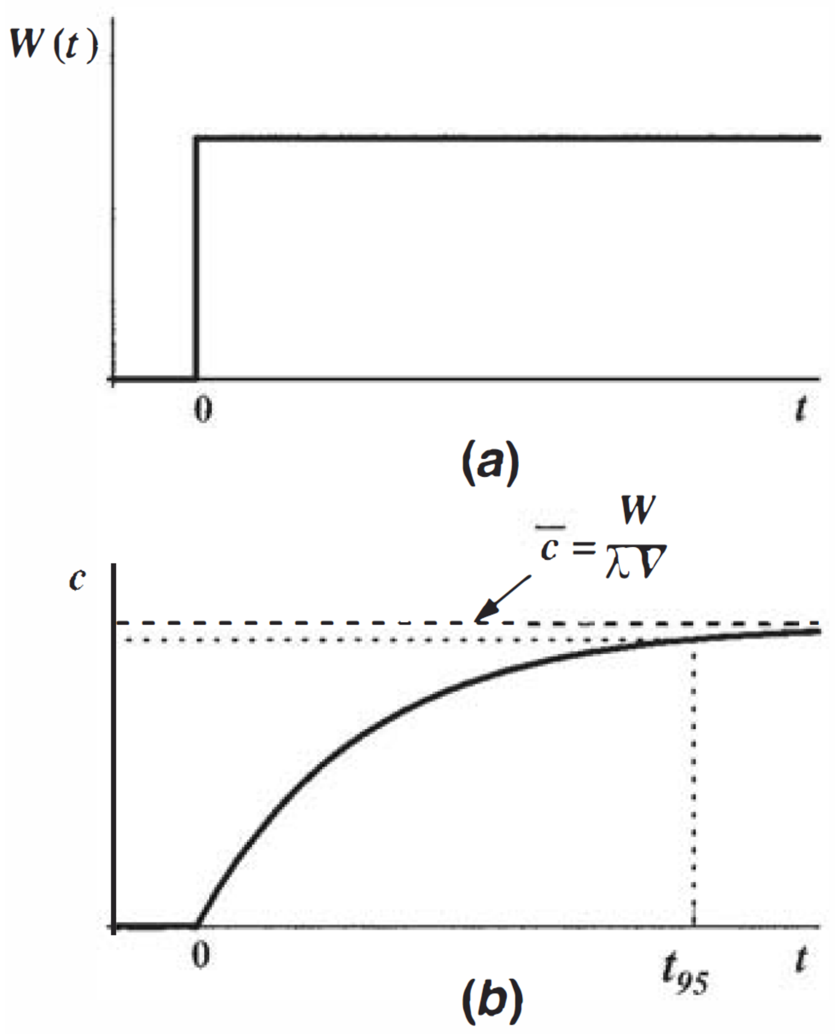


FIGURE 4.3
Plot of (*a*) loading and (*b*) response for step loading.

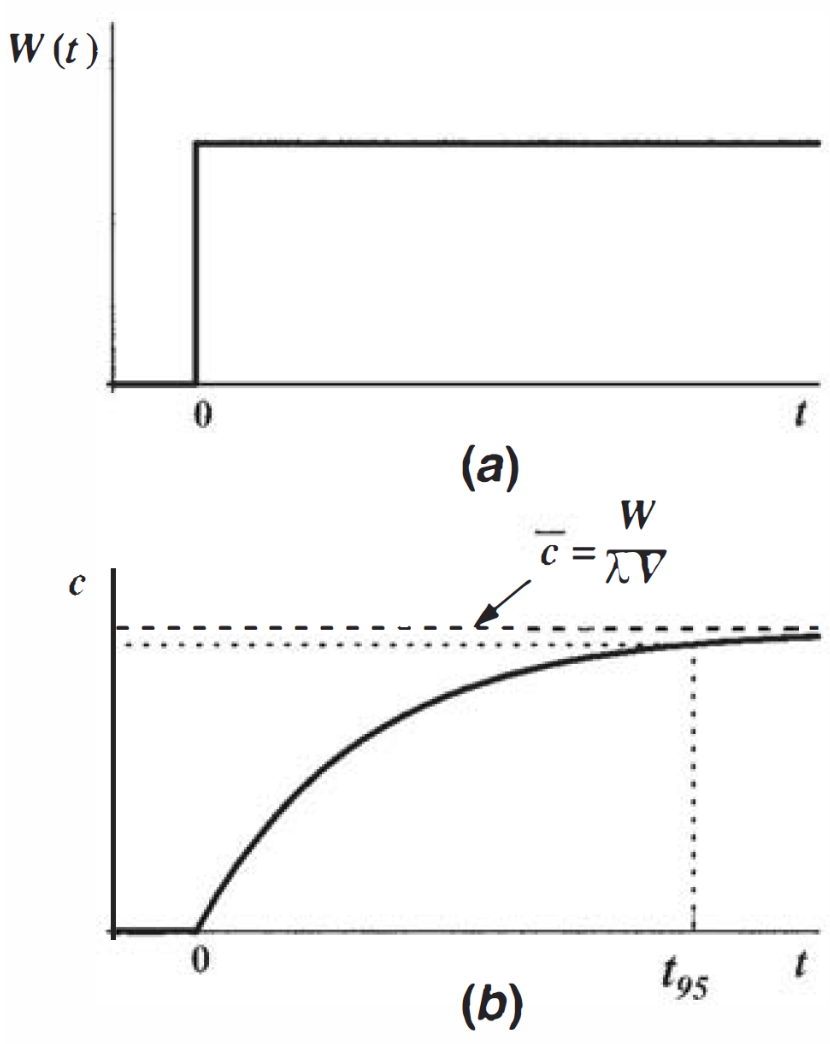
If a water body's loading is changed to a new constant level, then the forcing function is called a step input.

$$W(t) = 0 \qquad t < 0$$
$$W(t) = W \qquad t \geq 0$$

Where W(t) is the new constant level of loading (MT^{-1}). The particular solution is:

$$c = \frac{W}{\lambda V}(1 - e^{-\lambda t})$$

Step Loading (New Continuous Source)



$$c = \frac{W}{\lambda V}(1 - e^{-\lambda t})$$

At t=∞, the equation approaches steady state since the exponent becomes very small and we get:

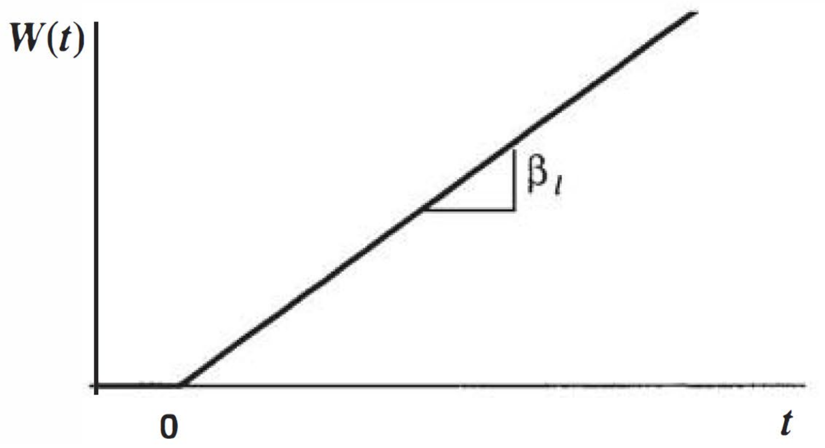
$$\bar{c} = \frac{W}{\lambda V}$$

which defines the height of the response and the response time defines its temporal extent ($t_{95} = \frac{3}{\lambda}$ if 95% chosen).

FIGURE 4.3
Plot of (*a*) loading and (*b*) response for step loading.

EXAMPLE 4.1. STEP LOADING. At time zero, a sewage treatment plant began to discharge 10 MGD of wastewater with a concentration of 200 mg L^{-1} to a small detention basin (volume $= 20 \times 10^4$ m^3). If the sewage decays at a rate of 0.1 d^{-1}, compute the concentration in the system during the first 2 wk of operation. Also determine the shape parameters to assess the ultimate effect of the plant.

Linear ("Ramp") Loading



Waste loading inputs can often be represented with a trend of a straight line.

$$W(t) = \pm\beta_l t$$

where β_l =rate of change or slope of the trend (MT^{-2}; $\pm$). The particular solution is:

$$c = \pm\frac{\beta_l}{\lambda^2 V}\left(\lambda t - 1 + e^{-\lambda t}\right)$$

Linear ("Ramp") Loading

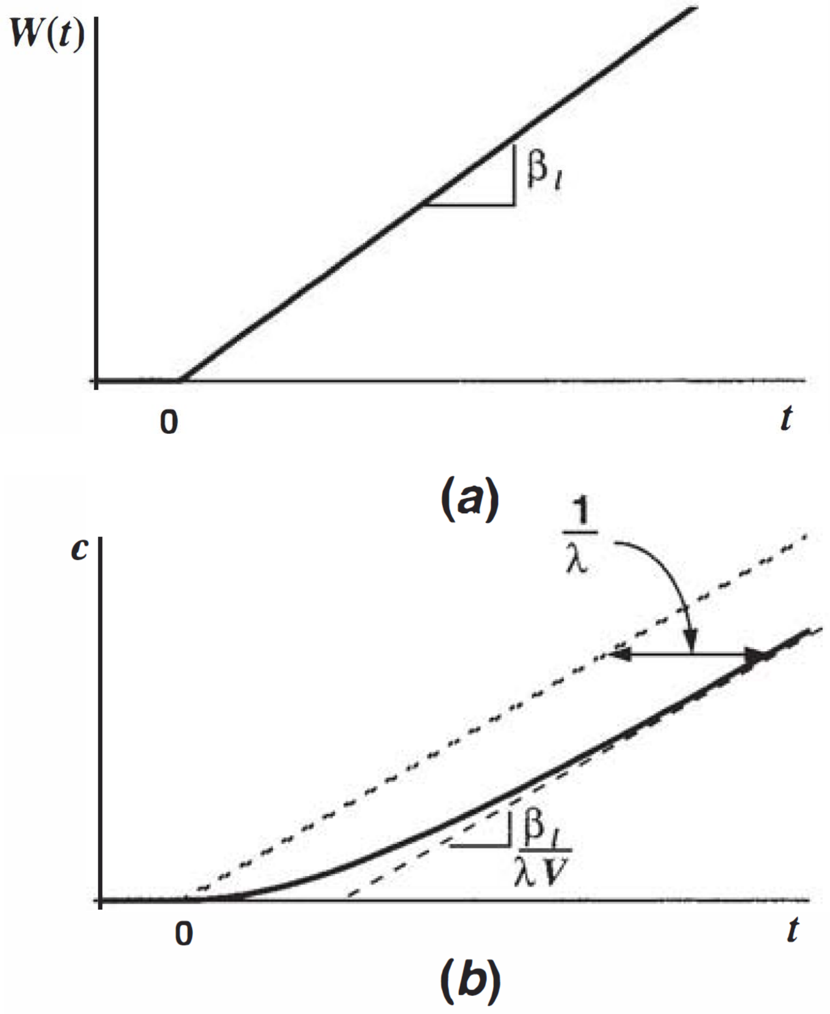


FIGURE 4.4
Plot of (a) loading and (b) response for a linearly increasing loading.

$$c = \pm\frac{\beta_l}{\lambda^2 V}\left(\lambda t - 1 + e^{-\lambda t}\right)$$

After initial start-up time, the solution becomes:

$$c = \frac{\beta_l}{\lambda V}\left(t - \frac{1}{\lambda}\right)$$

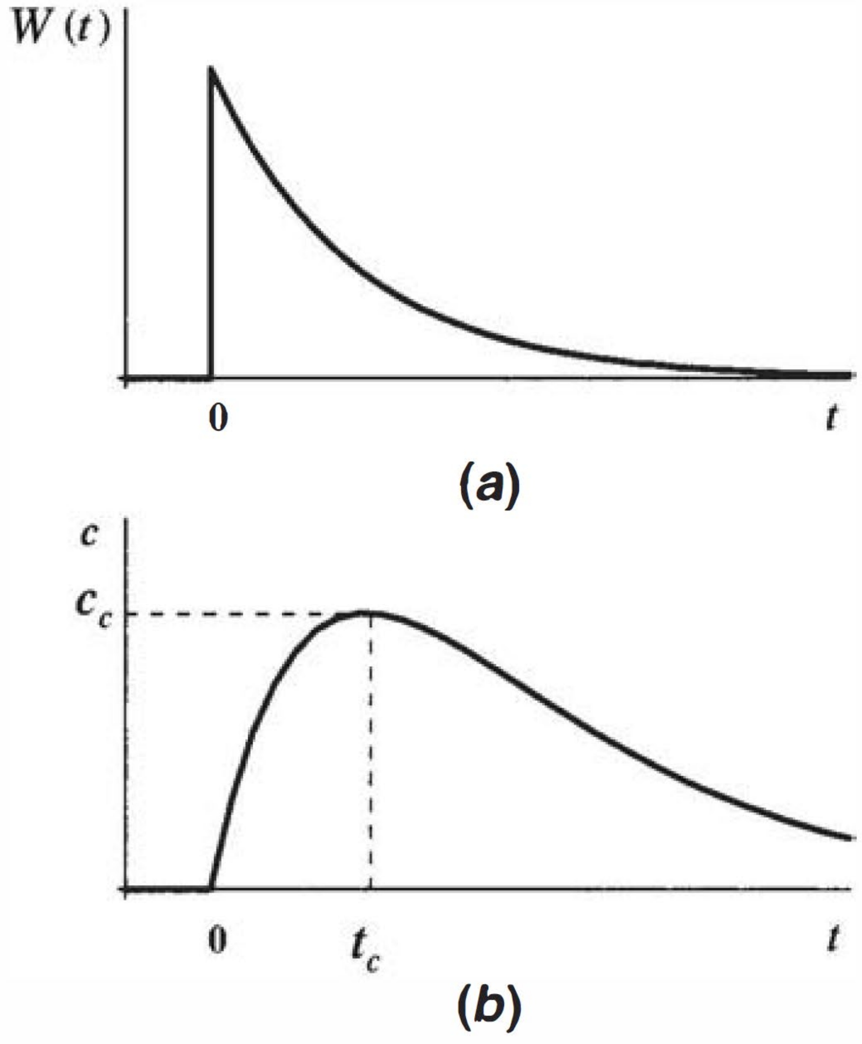
which is eventually increasing at a constant slope of:

$$\beta = \frac{\beta_l}{\lambda V}$$

with response lagging the input by

$$t_l = \frac{1}{\lambda}$$

Exponential Loading



Another way to characterize loading trends is an exponential function:

$$W(t) = W_e e^{\pm\beta_e t}$$

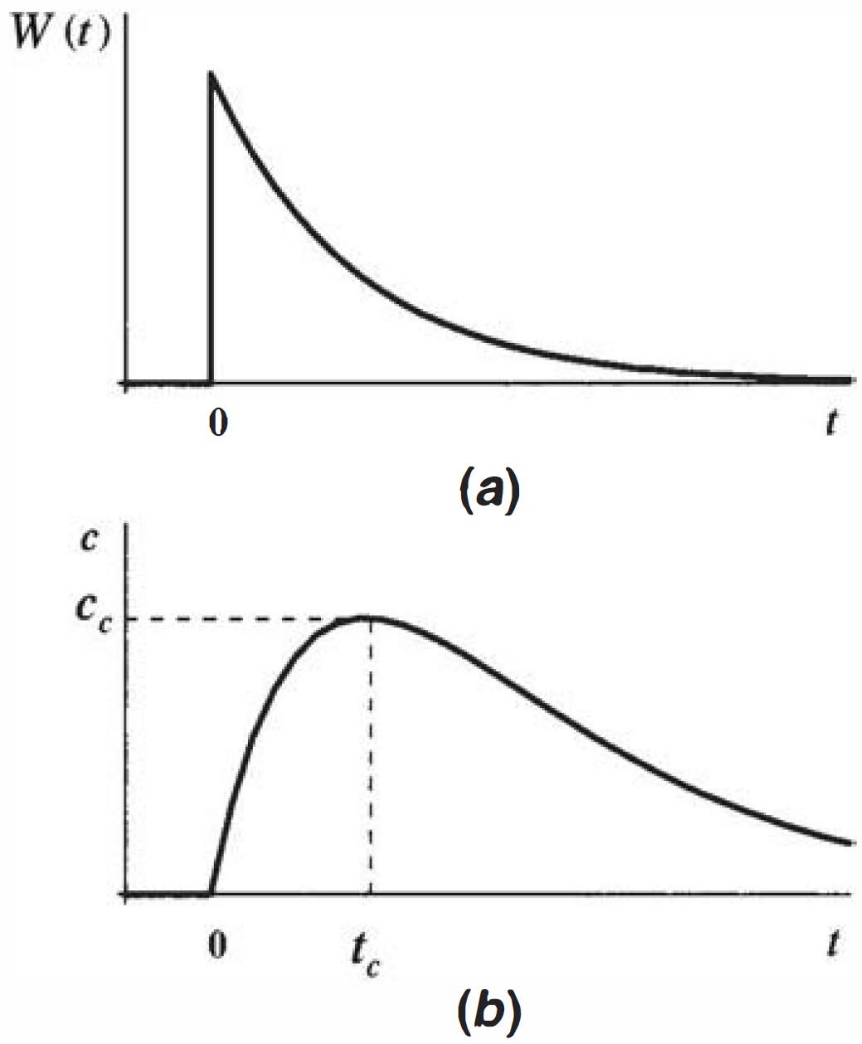
where W_e = a parameter that denotes the value at t = 0 (MT^{-1}) and β_e specifies the rate of growth (positive) or decay (negative) of the loading (T^{-1}).

The particular solution is:

$$c = \frac{W_e}{V(\lambda \pm \beta_e)}\left(e^{\pm\beta_e t} - e^{-\lambda t}\right)$$

FIGURE 4.5
Plot of (*a*) loading and (*b*) response for exponentially decaying loading.

Exponential Loading



The particular solution is:

$$c = \frac{W_e}{V(\lambda \pm \beta_e)}\left(e^{\pm\beta_e t} - e^{-\lambda t}\right)$$

From this we see two shape parameters: magnitude and time of the peak. Time of peak is found by differentiating above and rearranging:

$$t_c = \frac{\ln(\beta_e/\lambda)}{\beta_e - \lambda}$$

where "$_C$", is the "critical concentration. Magnitude is:

$$c_c = \frac{W_e}{\lambda V} e^{-\beta_e t_c}$$

FIGURE 4.5
Plot of (*a*) loading and (*b*) response for exponentially decaying loading.

Exponential Loading

Also since $\frac{dc}{dt} + \lambda c = \frac{W(t)}{V}$, and at peak $\frac{dc}{dt} = 0$, while

$$W(t) = W_e e^{\pm \beta_e t}$$

the "critical concentration" becomes:

$$c_c = \frac{W_e}{\lambda V} e^{-\beta_e t_c}$$

Further substitution of $t_c = \frac{\ln(\beta_e/\lambda)}{\beta_e - \lambda}$ into this equation yields:

$$c_c = \frac{W_e}{\lambda V} \left(\frac{-\beta_e}{\lambda} \right)^{\frac{\beta_e}{\lambda - \beta_e}}$$

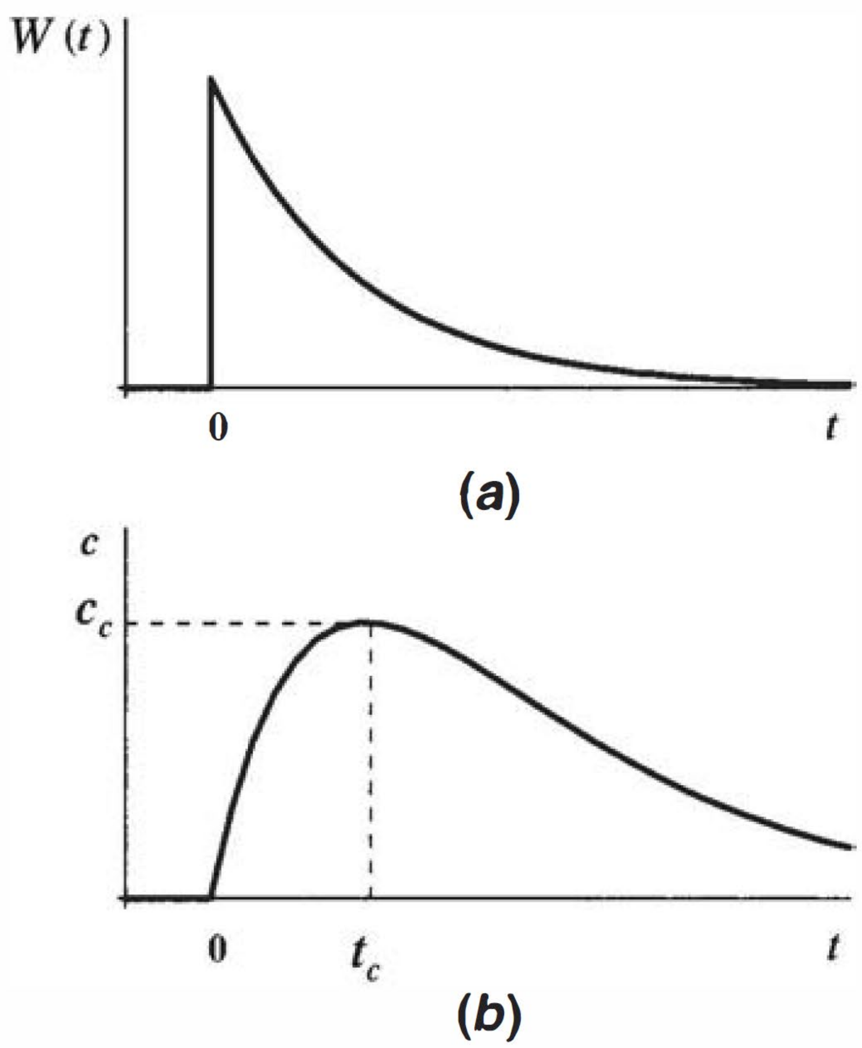


FIGURE 4.5
Plot of (*a*) loading and (*b*) response for exponentially decaying loading.

EXAMPLE 4.2. EXPONENTIAL FORCING FUNCTION. The following series of first-order reactions takes place in a batch reactor:

$$A \xrightarrow{k_1} B \xrightarrow{k_2}$$

Mass balance equations for these reactions can be written as

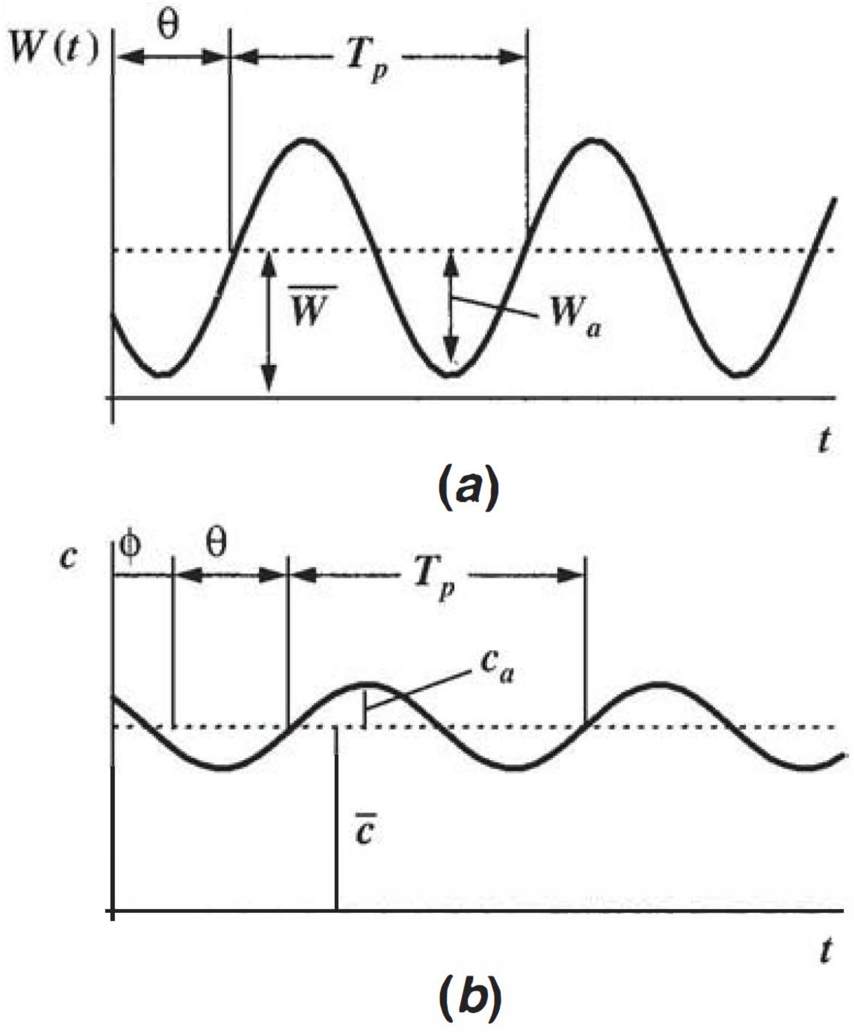
$$\frac{dc_A}{dt} = -k_1 c_A$$

and

$$\frac{dc_B}{dt} = k_1 c_A - k_2 c_B$$

Suppose that an experiment is conducted where $c_{A0} = 20$ and $c_{B0} = 0 \text{ mg L}^{-1}$. If $k_1 = 0.1$ and $k_2 = 0.2 \text{ d}^{-1}$, compute the concentration of reactant B as a function of time. Also, determine its shape parameters.

Sinusoidal Loading



The simplest periodic input is the sinusoidal function, which can be represented mathematically as:

$$W(t) = \overline{W} + W_a \sin(\omega t - \theta)$$

where $\overline{W}$ = mean loading (MT^{-1})

W_a = amplitude of the loading (MT^{-1})

θ = phase shift (radians)

ω = angular frequency of the oscillation (rT^{-1})

FIGURE 4.6
Plot of (*a*) loading and (*b*) response for the sinusoidal loading function. Note that a constant input is also shown in this illustration.

Sinusoidal Loading

ω = angular frequency of the oscillation (rT^{-1})

here the ω is defined as:

$$\omega = \frac{2\pi}{T_p}$$

Where T_p is the period of oscillation (T). Note that the simple frequency of oscillation can be computed as:

$$f = \frac{1}{T_p} \qquad (units\ of\ T^{-1}\ cycles)$$

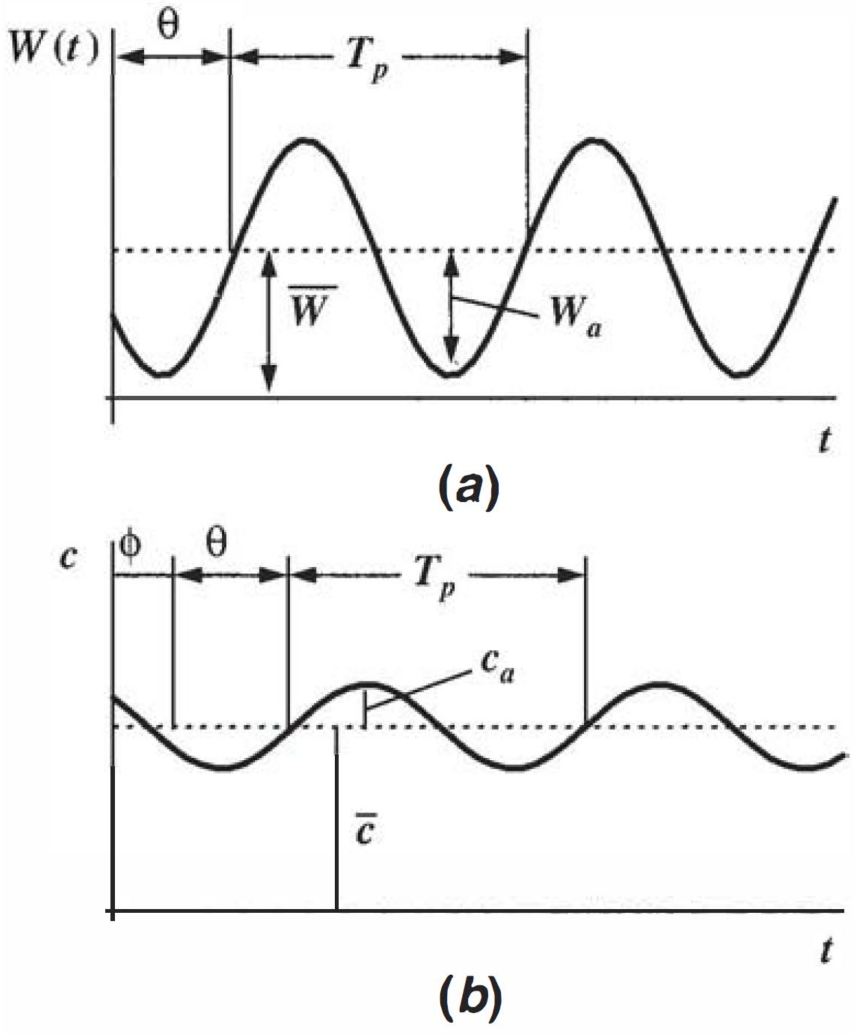
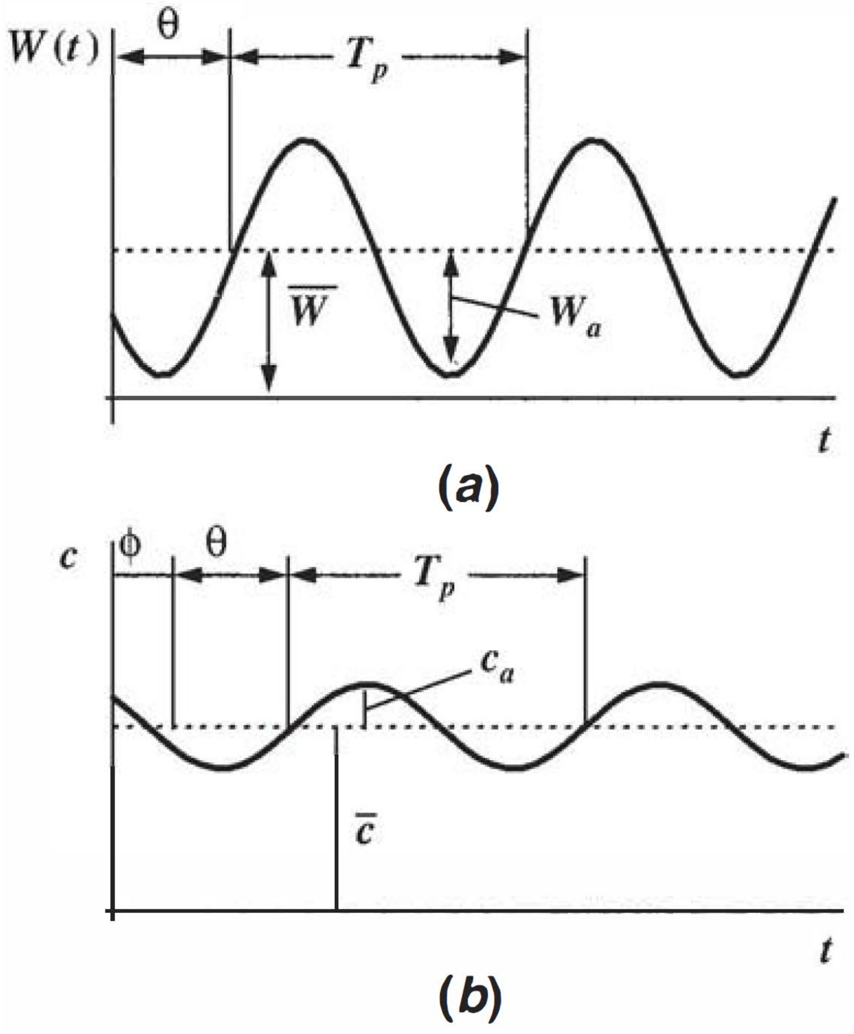


FIGURE 4.6
Plot of (a) loading and (b) response for the sinusoidal loading function. Note that a constant input is also shown in this illustration.

Sinusoidal Loading



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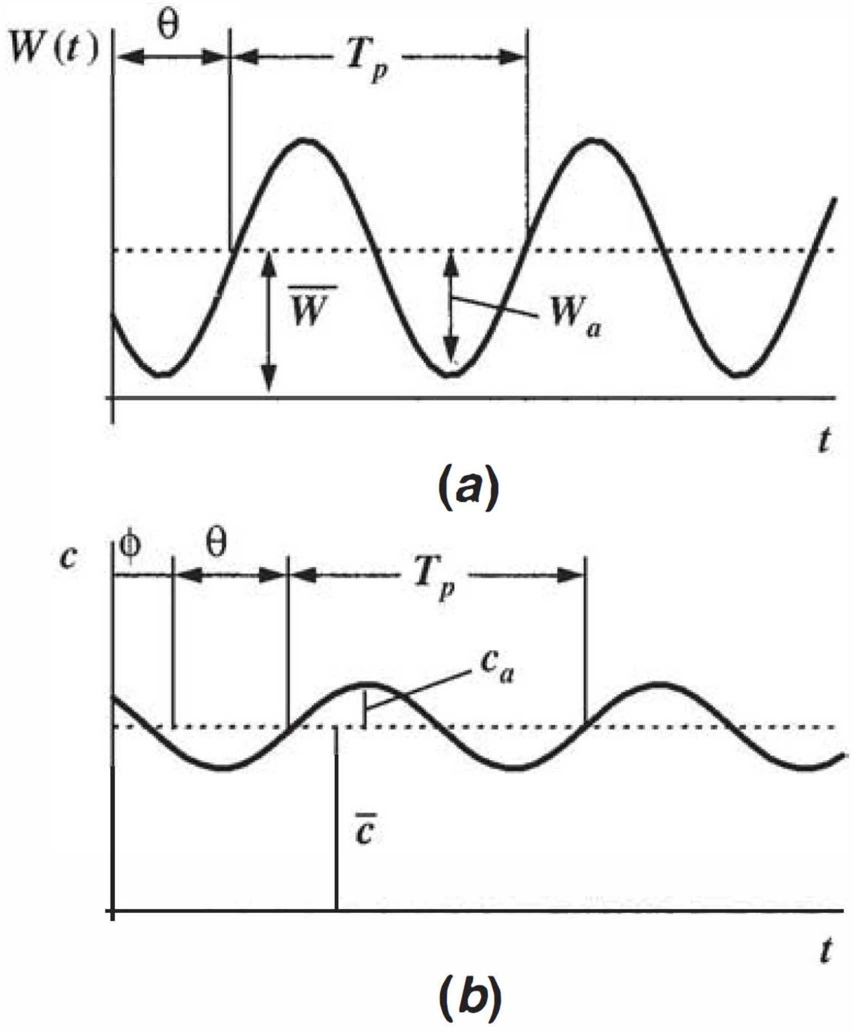
Sinusoidal Loading



The particular solution is:

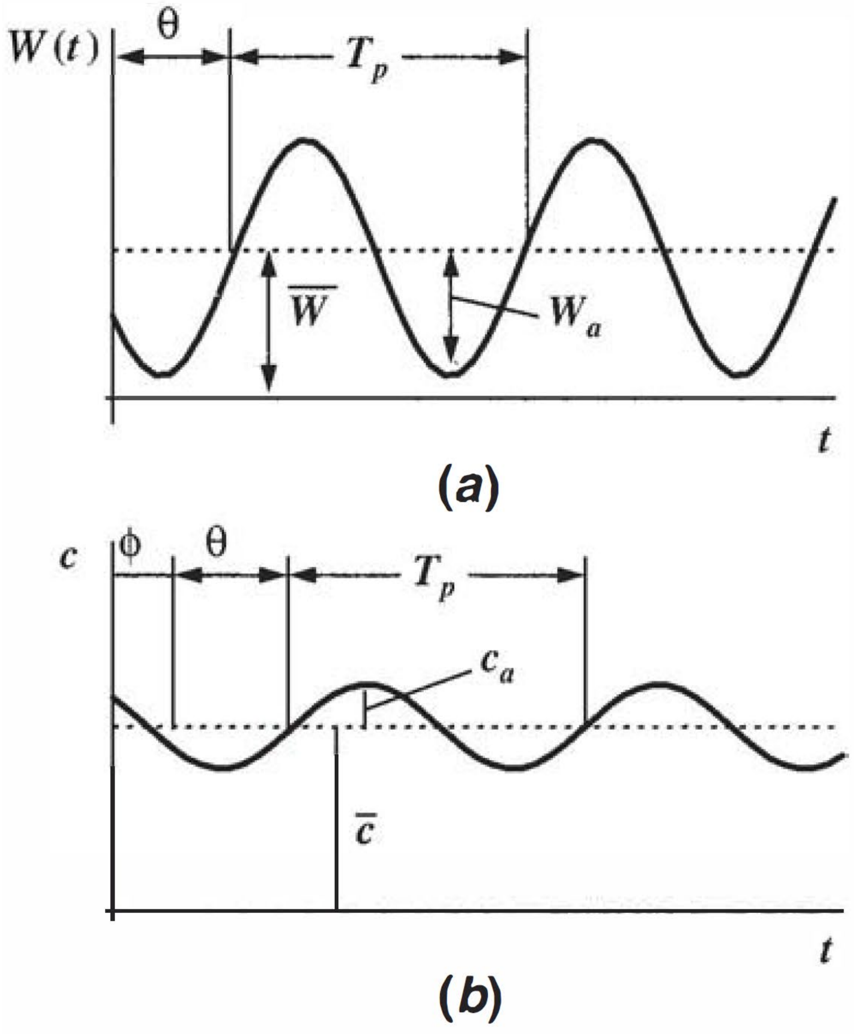
$$c = \frac{\overline{W}}{\lambda V}\left(1 - e^{-\lambda t}\right) + \frac{W_a}{V\sqrt{\lambda^2 + \omega^2}}\sin[\omega t - \theta - \varphi(\omega)] - \frac{W_a}{V\sqrt{\lambda^2 + \omega^2}}\sin[-\theta - \varphi(\omega)]e^{-\lambda t}$$

After a while this will be:

$$c = \frac{\overline{W}}{\lambda V}\left(1 - e^{-\lambda t}\right) + \frac{W_a}{V\sqrt{\lambda^2 + \omega^2}}\sin[\omega t - \theta - \varphi(\omega)]$$

FIGURE 4.6
Plot of (*a*) loading and (*b*) response for the sinusoidal loading function. Note that a constant input is also shown in this illustration.

Sinusoidal Loading



$$c = \frac{\bar{W}}{\lambda V}\left(1 - e^{-\lambda t}\right) + \frac{W_a}{V\sqrt{\lambda^2 + \omega^2}}\sin[\omega t - \theta - \varphi(\omega)]$$

Here there is an additional phase shift (in radians)

$$\varphi(\omega) = \tan^{-1}\left(\frac{\omega}{\lambda}\right)$$

which is one of the shape parameters. The other is the amplitude of the concentration response:

$$c_a = \frac{W_a}{V\sqrt{\lambda^2 + \omega^2}}$$

FIGURE 4.6
Plot of (*a*) loading and (*b*) response for the sinusoidal loading function. Note that a constant input is also shown in this illustration.