Streeter-Phelps: Distributed Sources
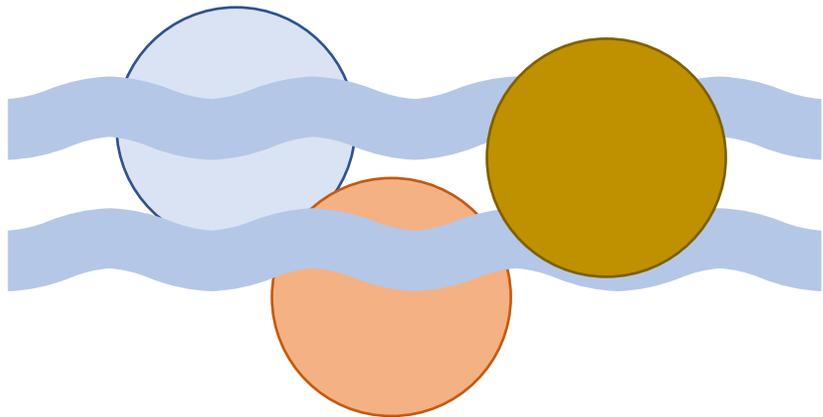
Parameterization of Distributed Sources

A steady-state mass balance for a distributed source could be written for a plug-flow with a uniform hydrology and geometry,

$$0 = -\frac{dc}{dt} - kc + S_d$$

where S_d = rate of the distributed source (g m^{-3} d^{-1}) and t = travel time for a plug-flow system. If $c = 0$ at $t = 0$, then:

$$c = \frac{S_d}{k}(1 - e^{-kt})$$

$S_d \, (M \, L^{-3} \, T^{-1})$

$x = Ut$

0

(a)

c

$\dfrac{S_d}{k}$

t

(b)
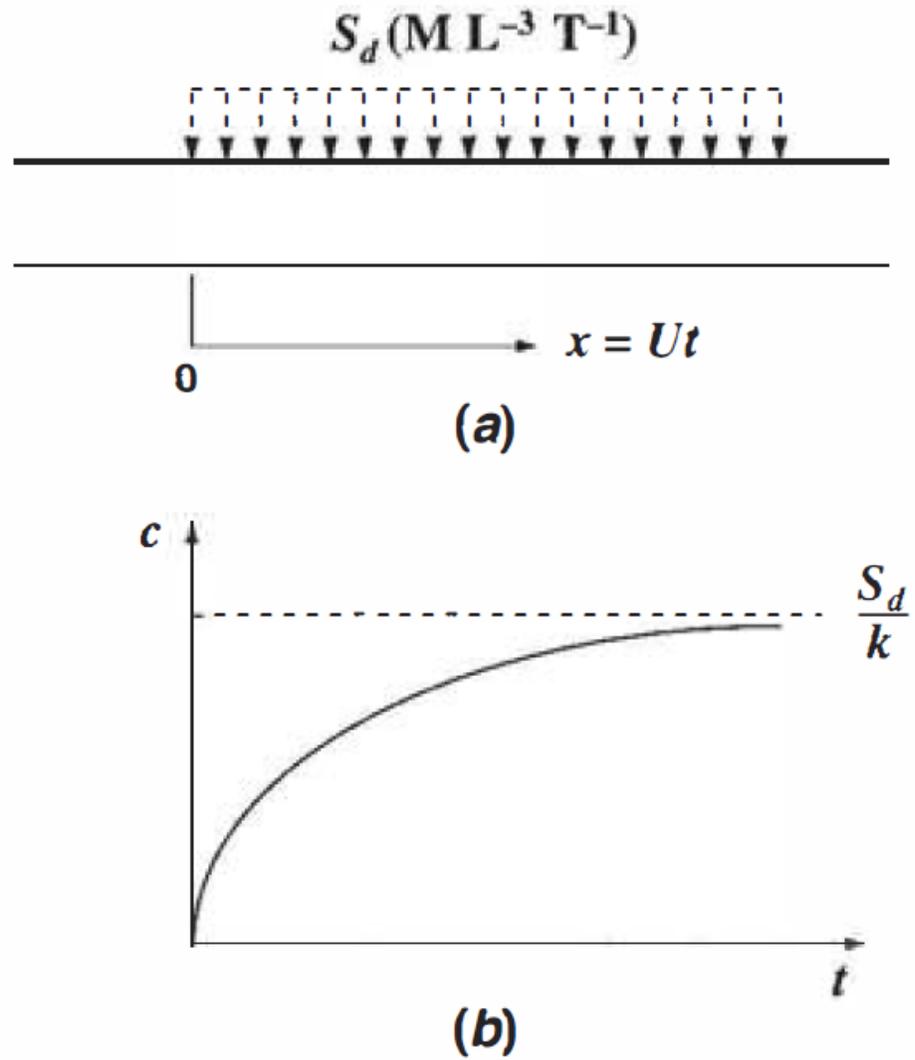
FIGURE 22.1
(a) Uniform distributed source and (b) resulting response if the substance decays with first-order kinetics.

Parameterization of Distributed Sources

The best way to parameterize the inputs would be per length of stream, $S_d^{''}$ $(ML^{-1}T^{-1})$. Such rate is converted to the proper volumetric units by:

$$S_d = S_d^{''} \frac{L}{V} = \frac{S_d^{''}}{A_c}$$

where L = total length

V = volume

A_c = cross-sectional area of the reach being loaded

$S_d^{''}$ (M L^{-1} T^{-1})
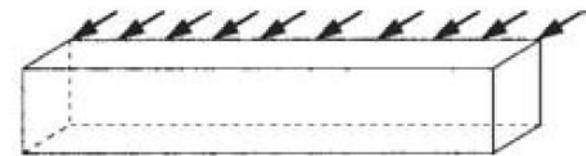
(a) Line load

Parameterization of Distributed Sources

The best way to parameterize such inputs would be as a flux S_d' $(ML^{-2}T^{-1})$. This mass balance would be :

$$S_d = S_d' \frac{A_s}{V} = \frac{S_d'}{H}$$

where H = stream depth.

S_d' (M L^{-2} T^{-1})

(b) Areal load

S_d (M L^{-3} T^{-1})
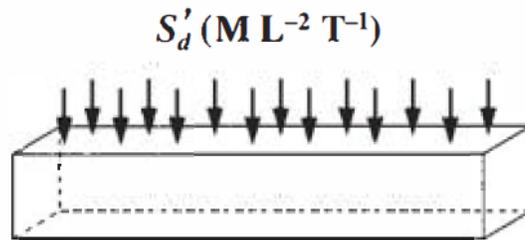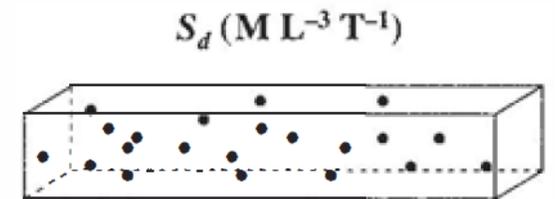
(c) Volumetric load

No-Flow Sources

No-Flow Sources (BOD)

For BOD introduced to a system without adding significant flow are interactions with porous media. The mass balance would be written as:

$$0 = -\frac{dL}{dt} - k_r L + S_L$$

where S_L = rate of the BOD distributed source (g m^{-3} d^{-1}).

(BOD)
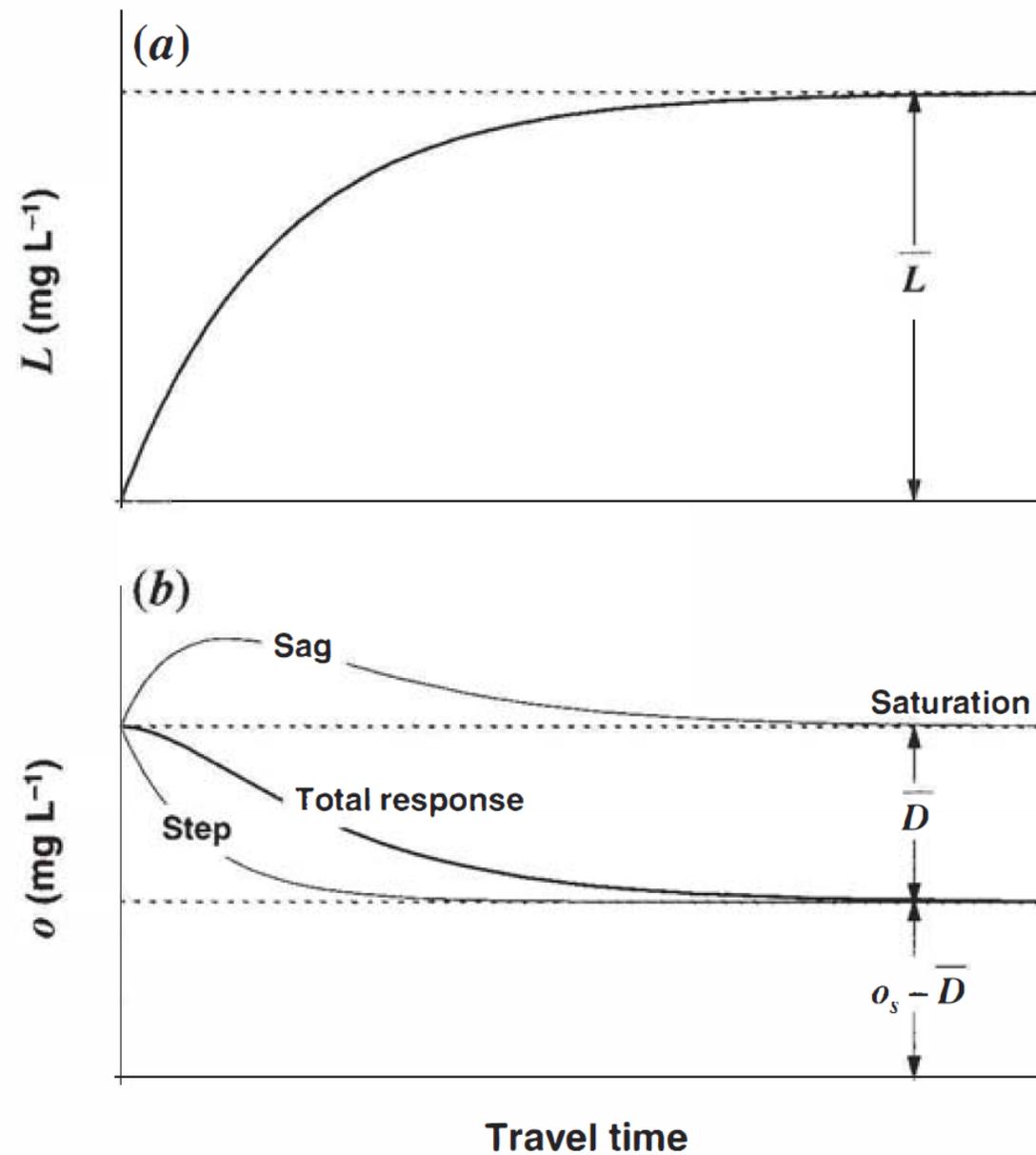
FIGURE 22.3
Plot of the (a) BOD and (b) oxygen responses due to a distributed BOD loading.

(BOD)

The solution if $L = 0$ at $t = 0$, would be:

$$L = \frac{S_L}{k_r}(1 - e^{-k_r t})$$

Thus the BOD increases in a step-response fashion and asymptotically approaches a steady-state level $\bar{L}$, where the BOD gain is balanced by removal:

$$\bar{L} = \frac{S_L}{k_r}$$

A mass balance for oxygen deficit can be written as:

$$0 = -\frac{dD}{dt} - k_a D + \frac{k_d}{k_r} S_L (1 - e^{-k_r t})$$
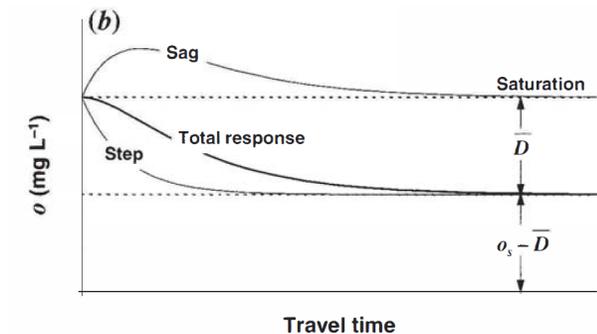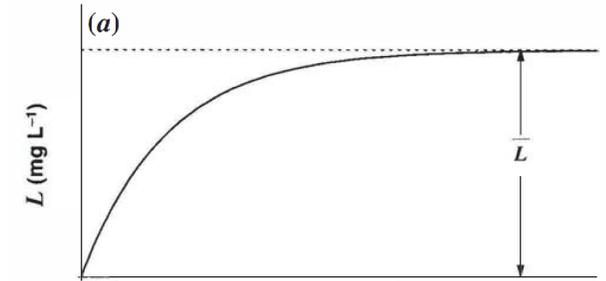
FIGURE 22.3
Plot of the (a) BOD and (b) oxygen responses due to a distributed BOD loading.

(BOD)

Thus the BOD would act as a forcing function contribution to the deficit. For the case D= 0 at t=0, this balance

$$0 = -\frac{dD}{dt} - k_a D + \frac{k_d}{k_r} S_L (1 - e^{-k_r t})$$

can be solved for:

$$D = \frac{k_d S_L}{k_r k_a}(1 - e^{-k_a t}) - \frac{k_d S_L}{k_r(k_a - k_r)}(e^{-k_r t} - e^{-k_a t})$$

The deficit increases in a step-response fashion, steady-state

$$\bar{D} = \frac{k_d S_L}{k_r k_a}$$

Dissolved Oxygen

The most commonly employed distributed load, used to simulate the effect of plants and sediment oxygen demand can be written as a mass balance.

$$0 = -\frac{dD}{dt} - k_a D - P + R + \frac{S_B'}{H}$$

where P and R = volumetric rates of plant photosynthesis and respiration, respectively (g m^{-3} d^{-1})

S_B' = areal rate of sediment oxygen demand (g m^{-2} d^{-1})

H = depth (m)

Dissolved Oxygen

The solution, if $D = 0$ at $t = 0$, would be

$$\overline{D} = \frac{-P + R + (S'_B/H)}{k_a}(1 - e^{-k_a t})$$

Thus the deficit increases in a step-response fashion, approaching a steady-state level $\overline{D}$, where net deficit gain is balanced by reaeration losses:

$$\overline{D} = \frac{-P + R + (S'_B/H)}{k_a}$$

Total Streeter-Phelps Model

We've developed formulations for both point and non-point sources of BOD and oxygen.
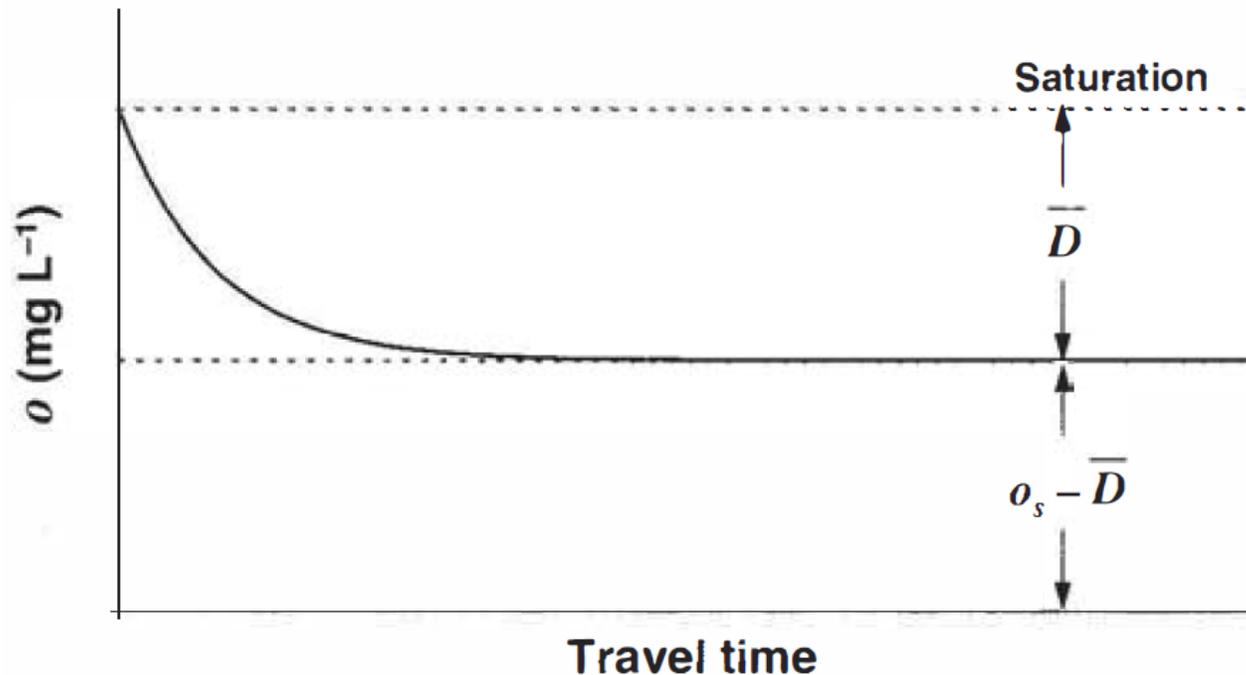
FIGURE 22.4
Plot of the oxygen response due to a distributed oxygen loading.

Total Streeter-Phelps Model

We've developed formulations for both point and non-point sources of BOD and oxygen.

Oxygen deficit is:

$$L = L_0 e^{-k_r t} + \frac{S_L}{k_r}(1 - e^{-k_r t})$$

Point   Distributed

$$D = D_0 e^{-k_a t} + \frac{k_d L_0}{k_a - k_r}(e^{-k_r t} - e^{-k_a t}) + \frac{-P + R + (S_B'/H)}{k_a}(1 - e^{-k_a t})$$

Point deficit   Point BOD   Distributed deficit

$$+ \frac{k_d S_L}{k_r k_a}(1 - e^{-k_a t}) - \frac{k_d S_L}{k_r(k_a - k_r)}(e^{-k_r t} - e^{-k_a t})$$

Distributed BOD

Analytical Solutions

A mass balance for a diffuse source that contributes both flow and mass can be written as:

$$\frac{\partial(A_c c)}{\partial t} + \frac{\partial(Qc)}{\partial x} = \frac{dQ}{dx} c_d - kA_c c$$

where A_c = cross-section area (m²)

c = concentration (mg L^{-1})

t = time (s)

Q = flow rate (m³ s^{-1})

x = distance (m)

c_d = concentration of the diffuse source (mg L^{-1})

k = first-order decay rate (s^{-1})

Analytical Solutions

At steady state this becomes:

$$0 = -\frac{d(Qc)}{dx} + \frac{dQ}{dx}c_d - kA_c c$$

O'Conner developed both steady-state and time-variable solutions for dissolved solids ($k = 0$). O' Conner idealized flow increases as an exponential function,

$$Q = Q_0 e^{q'x}$$

where Q_0 = flow at $x = 0$ (m$_3$ d^{-1}) and q' = exponential rate of flow increase (m^{-1}). This equation can be substituted above to yield:

$$\frac{dc}{dx} + q'c = q'c_d$$

Analytical Solutions

$$\frac{dc}{dx} + q'c = q'c_d$$

Applying the boundary condition $c = c_0$ at $x = 0$, the obtained solution is:

$$c = c_0 e^{-q'x} + c_d(1 - e^{-q'x})$$
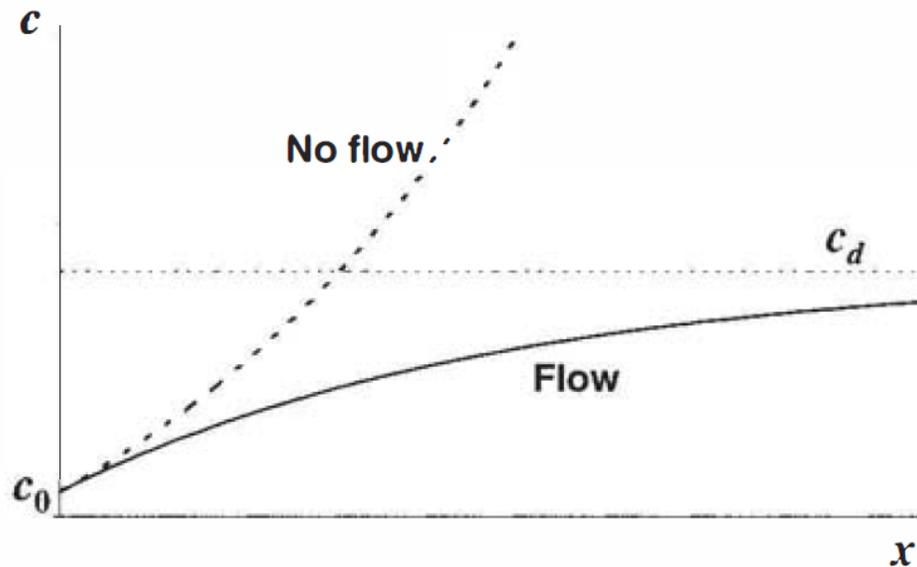
FIGURE 22.5
Plot of concentration versus distance where nonpoint sources add flow and a dissolved conservative pollutant to a stream. For this case the in-stream concentration asymptotically approaches the concentration of the diffuse source. In addition we also show how the concentration would increase exponentially for the case where the diffuse flow contributes mass but not flow.

Analytical Solutions

The addition of a reaction term complicates the analysis because we must now consider how the diffuse flow affects the stream's hydrogeometric parameters. We have to account for the flow additions impacting both velocity and cross-sectional area
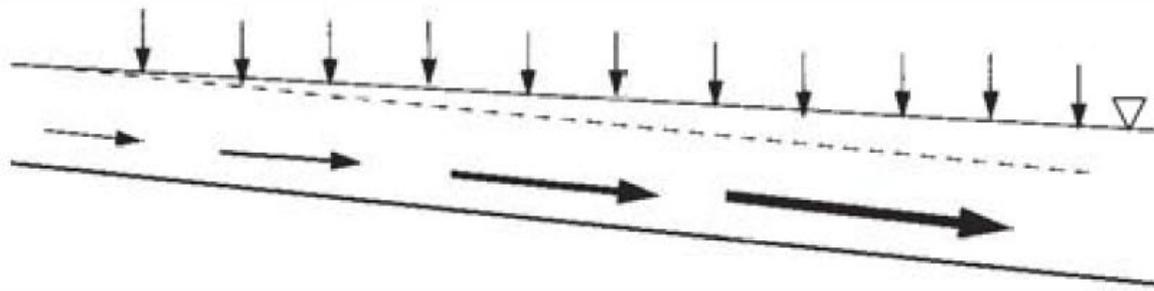
FIGURE 22.6
Distributed sources of flow cause water in a channel to both move faster and run deeper.

Analytical Solutions

To incorporate these effects into our model we employ a linear function to represent the flow increases,

$$Q = Q_0 + qx$$

where q = a constant parameterizing the rate of the linear increase (m²d⁻¹). Substituting this into ($0 = -\dfrac{d(Qc)}{dx} + \dfrac{dQ}{dx}c_d - kA_cc$) gives:

$$0 = -(Q_0 + qx)\frac{dc}{dx} - (q + kA_c)c + qc_d$$
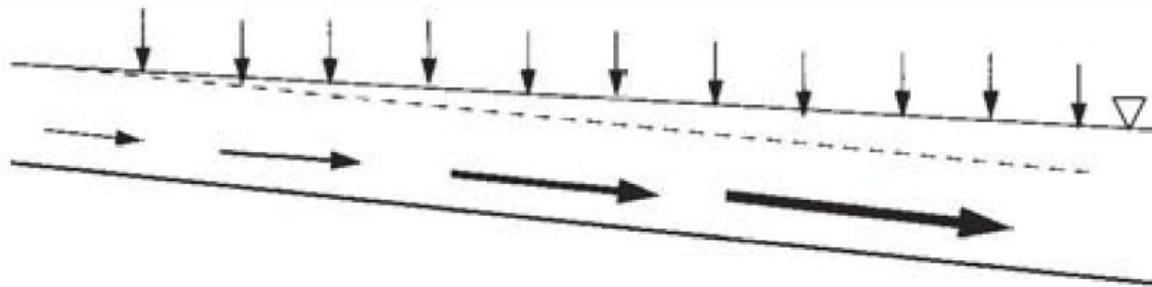
FIGURE 22.6
Distributed sources of flow cause water in a channel to both move faster and run deeper.

Analytical Solutions

$$0 = -(Q_0 + qx)\frac{dc}{dx} - (q + kA_c)c + qc_d$$

This is a linear rise, which is reasonable (although exponential is possible as well). Since it is over short distances, the numerical approach described next will be using linearly increasing flow.

A solution first to the extreme case :

$$c = c_0 \left(\frac{U_0}{U_o + vx}\right)^{\frac{k+v}{v}} + c_d \frac{v}{v + k}\left[1 - \left(\frac{U_0}{U_o + vx}\right)^{\frac{k+v}{v}}\right]$$

where

$$v = \frac{q}{A_{c0}}$$

in which A_{c0} = initial area (m^2).

Analytical Solutions

Then for the steady-state case :

$$c = \left[c_0 e^{-\frac{k}{U_0}x} + c_d \frac{v}{k}(1 - e^{-\frac{k}{U_0}x}) \right]\left(\frac{U_0}{U_o + vx} \right)$$

As can be seen the constant velocity result eventually goes to zero. In contrast the constant-area solution asymptotically approaches a constant concentration:

$$c = c_d \frac{v}{v + kA_{c0}}$$
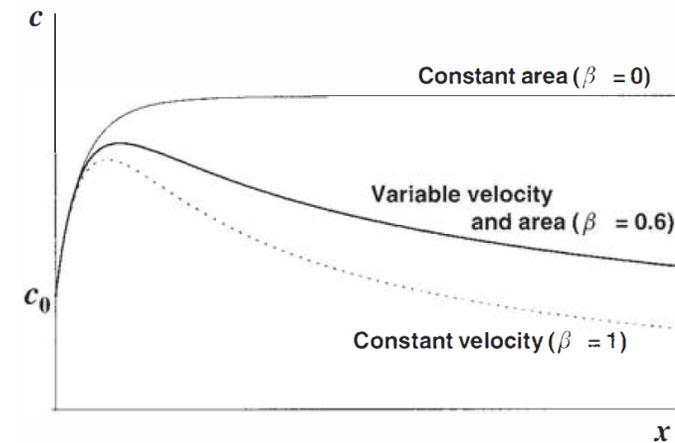
FIGURE 22.7
Plot of concentration versus distance for three cases: constant area, area and velocity varying according to Manning's equation, and constant velocity.

Analytical Solutions

Clearly the correct result for realistic cases lies between the two extremes. To quantify this intermediate case we assume that the Manning Equation provides an adequate representation of the momentum balance for the steady-flow in the channel

$$Q = \frac{C_0}{n} A_c R^{2/3} S_e^{1/2}$$

where C_0 = constant (1: SI, 1.486: BG)

n = Manning's roughness coefficient

R = channel's hydraulic radius (m or ft)

P = wetted perimeter (m or ft)

S_e = slope of the channel (m/m or ft/ft)
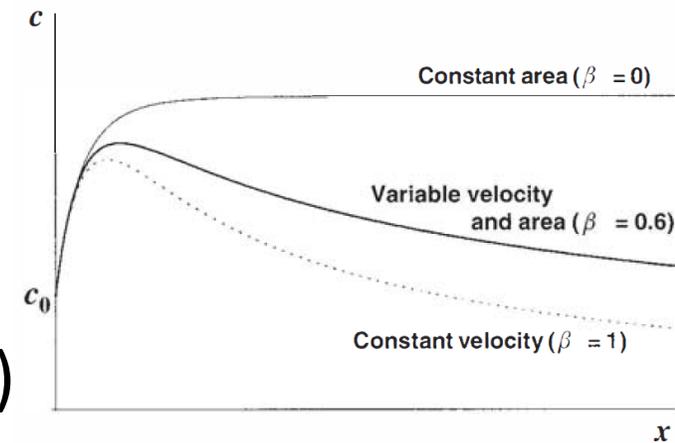
FIGURE 22.7
Plot of concentration versus distance for three cases: constant area, area and velocity varying according to Manning's equation, and constant velocity.

Analytical Solutions

Assuming metric units, we can solve Manning's equation for

$$A_c = \alpha Q^{\beta}$$

where β = 3/5 and for the case of a wide, shallow rectangular channel,

$$\alpha = \left(\frac{nB^{\frac{2}{3}}}{\sqrt{S_0}} \right)^{3/5}$$

where B = channel width. For this simple case, because α is constant, a simple relationship between flow and area is established.

The parameter β specifies the relationship between area and velocity. Values of β vary from 0 to 1.

Analytical Solutions
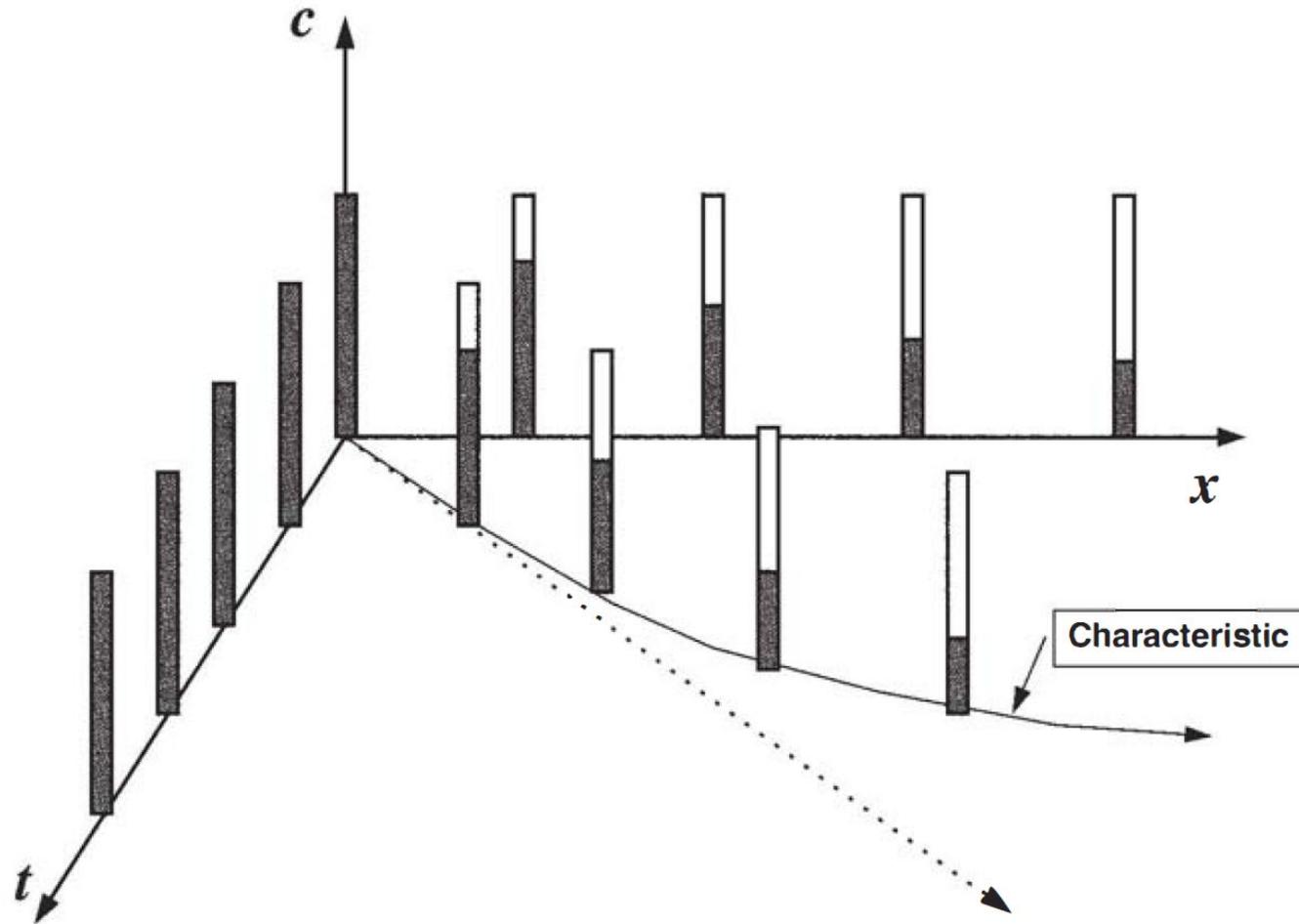
FIGURE 22.8
Graphical depiction of the method of characteristics for a plug-flow system where flow increases with distance.

Analytical Solutions

We apply the original partial differential equation to express as a coupled pair of ordinary differential equations:

$$\frac{dx}{dt} = U$$

$$\frac{dc}{dt} = -kc + \frac{q}{A_c}c_d - \frac{q}{A_c}c$$
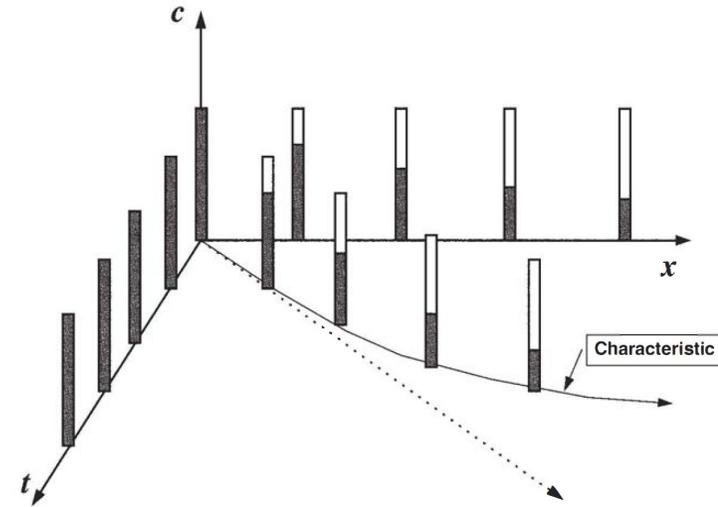
The first equation describes the characteristic trajectory. The second equation shows how the pollutant changes temporally as it moves along the characteristic.

Analytical Solutions

Substituting into previous equations yields:

$$\frac{dx}{dt} = \frac{1}{\alpha}(Q_0 + qx)^{(1-\beta)}$$

$$\frac{dc}{dt} = \frac{q(c_d - c)}{\alpha(Q_0 + qx)^{(1-\beta)}} - kc$$

Applying appropriate initial conditions (t=0, x=0, Q=Q0)),

we can solve

$$x = \left[\left(\frac{\beta q}{\alpha}t + Q_0^\beta\right)^{1/\beta} - Q_0\right]\frac{1}{q}$$
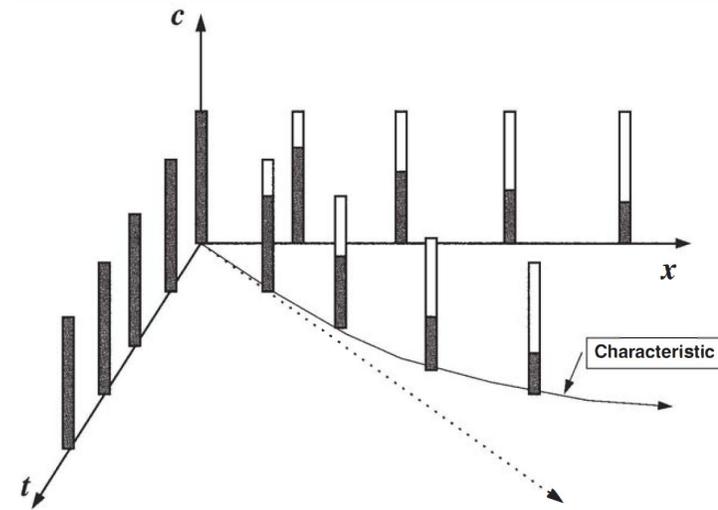
Numerical Method

A computer-oriented numerical method is better suited for general application, beyond the limited application of the analytical approach.

The following is similarly employed in QUAL 2E model.

Equal-length computational elements represent reaches of uniform geometry river sections. Total diffuse flow Q_d is established for each reach, each element receiving Q_d/n flow. If each segment is Δx, then Q_d and q are related by:

$$Q_e = \frac{Q_d}{n} = q\Delta x$$

where Q_e = incremental flow for each element ($m^3 \, d^{-1}$).
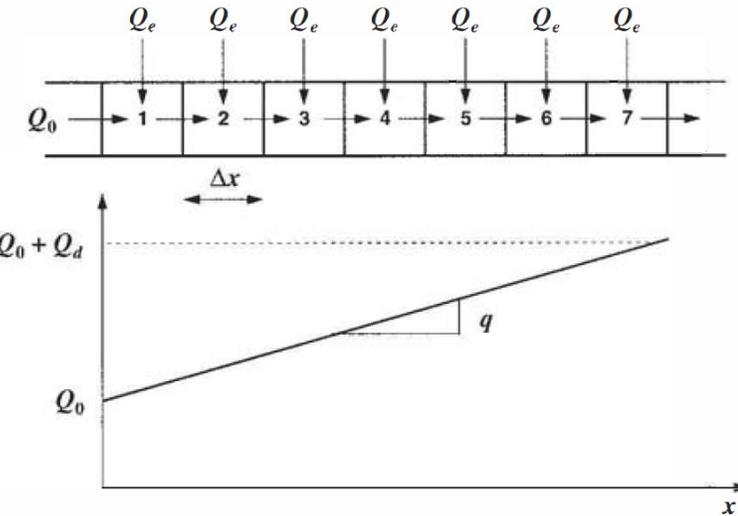
FIGURE 22.9
Plot of a finite-difference scheme accounting for a uniformly distributed load.

Numerical Method

$$Q_e = \frac{Q_d}{n} = q\Delta x$$

where Q_e = incremental flow for each element ($m^3 \ d^{-1}$).

FIGURE 22.9
Plot of a finite-difference scheme accounting for a uniformly distributed load.

Transport calculation is performed element-by-element,

starting at the element farthest upstream. First, simple flow balances are computed for the cell: e.g. in the first element:

$$Q_1 = Q_0 + Q_e$$

where Q_1 = outflow from element 1.

Numerical Method

FIGURE 22.9
Plot of a finite-difference scheme accounting for a uniformly distributed load.

Next for a wide rectangular channel, the flow can be used to determine the element volume:

$$V = \Delta x A_c = \Delta x \alpha Q^\beta$$

The mass balance can be developed for example for BOD and oxygen.

$$0 = Q_{i-1}L_{i-1} - Q_i L_i + Q_e L_{d,i} - k_{r,i}V_i L_i$$

$$0 = Q_{i-1}o_{i-1} - Q_i o_i + Q_e o_{d,i} - k_{d,i}V_i L_i + k_{a,i}V_i\left(o_{s,i} - o_i\right) + P_i V - R_i V - S'_{b,i}A_s$$

The balances can be solved in sequence

Numerical Method

FIGURE 22.9
Plot of a finite-difference scheme accounting for a uniformly distributed load.

The balances can be solved in sequence

$$L_i = \frac{Q_{i-1}L_{i-1} + Q_e L_{d,i}}{Q_i + k_{r,i}V_i}$$

$$o_i = \frac{Q_{i-1}o_{i-1} + Q_e o_{d,i} - k_{d,i}V_i L_i + k_{a,i}V_i o_{s,i} + P_i V - R_i V - S'_{b,i}A_s}{Q_i + k_{a,i}V_i}$$

Numerical Method

A numerical dispersion is manifested that is approximately :

$$E_n = \frac{\Delta x}{2} U$$

FIGURE 22.10
Plot of concentration versus distance for BOD and DO for distributed sources with flow.

Real streams exhibit dispersion. Consequently the physical dispersion provides an upper bound on the spatial step:

$$\Delta x = \frac{2E}{U}$$